

Cinderella, Quadrilaterals and Conics

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Abstract

We study quadrilaterals inscribed and circumscribed about conics. Our research is guided by experiments in software *Cinderella*. We extend the known results in projective geometry of conics and show how modern mathematical software brings new ideas in pure and applied mathematics. Poncelet theorem for quadrilaterals is proved by elementary means together with Poncelet's grid property.

1 Introduction

This paper is one in the serial of our forthcoming papers in geometry of curves, combinatorics and dynamic systems. The use of software *Cinderella* is common for all of them and our aim is to show that the good software is more than a box with nice examples and calculations. The smart use could lead us not only to discovering new results, but it gives the complete and correct proofs! In this sense *Cinderella* could go beyond the limits of geometry of conics and mechanical experiments, even to the curves of higher degree and abstract combinatorics, geometry and topology.

The positive experience with *Cinderella* in the paper *Illumination of Pascal's Hexagrammum and Octagrammum Mysticum* by Baralić and Spasojević, [1] encouraged us to continue the research. The problems we study are strongly influenced by very inspirative paper *Curves in Cages: an Algebro-geometric Zoo* of Gabriel Katz printed in American Mathematical Monthly, [10]. Many important questions in dynamical systems and combinatorics have their equivalents in the terms of algebraic curves. Richard Schwartz and Serge Tabachnikov in [21] asked for the proof of Theorem 4.c. They found the theorem studying the pentagram maps, introduced in [19]. This is still open hypothesis and could be reformulated in the question about curves.

We have not found the proof for Schwartz and Tabachnikov Theorem 4.c but during recent work we discovered new interesting facts about quadrilaterals inscribed and quadrilaterals circumscribed about conic. Theorems about quadrilaterals and conics are usually known like degenerate cases of Pascal and Brianchon theorems. In [1] Baralić and Spasojević proved some new results about two quadrilaterals inscribed

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in a conic. However, in this paper we study more complicated structures involving both tangents at the vertices and the side lines of quadrilateral. We start from the degenerate form of Pascal and Brianchon theorems for the quadrilateral and then we discover new interesting points, conics and loci.

The objects are studied by elementary means. Some of the results are in particular the corollary of Great Poncelet Theorem for the case when n -gon is quadrilateral. Here we give the short proof for this case. Some special facts about this special case are explained as well.

Finally, we compare two theorems - Mystic Octagon theorem for the case of two quadrilaterals and Poncelet Theorem for the quadrilaterals. Both of them have in common that they state that certain 8 points coming from two quadrilaterals inscribed in a conic lie on the same conic. While the first one is pure algebro-geometric fact, the latter involves much deeper structure of the space and can not be seen naturally as the special case of the first. Thus, we could not find 'Theorem of all theorems for conics in projective geometry' and elementary surprises in projective geometry like those in [21] could come as the special case of different general statements.

2 From Pascal to Brocard Theorem

In this section we show how Pascal theorem for hexagon (1639) inscribed in a conic degenerates to Brocard theorem for the quadrilateral inscribed in a circle. All results here are well known and are part of the standard olympiad problem solving curriculum, but our aim is to illustrate the power of degeneracy tool and prepare the background for the next sections.

Lemma 2.1. *Let $ABCD$ be a quadrilateral inscribed in a conic C and let M be the intersection point of the lines AD and BC , N be the intersection point of the lines AB and CD , P be the intersection point of the tangents to C at A and C , and Q be the intersection point of the tangents to C at B and D . Then, the points M , N , P and Q are collinear (see Figure 1).*

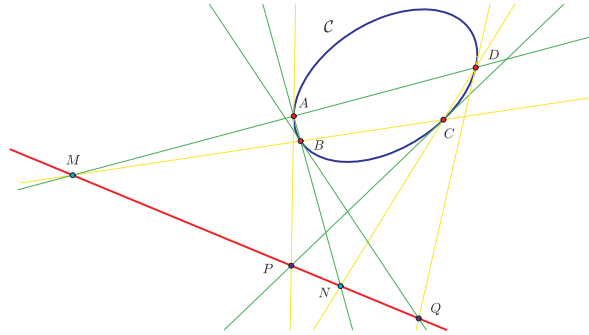


Figure 1: Lemma 2.1

Proof: Apply Pascal theorem to degenerate hexagon $AABCCD$ and we get the points M , N and P are collinear. Apply Pascal theorem to degenerate hexagon $ABBCD$ and we get the points M , N and Q are collinear. \square

Dual statement to Lemma 2.1 is the following:

Lemma 2.2. *Let conic \mathcal{C} touch the sides AB , BC , CD and DA of a quadrilateral $ABCD$ in the points M , N , P and Q , respectively. Then the lines AC , BD , MP and NQ pass through the same point O (see Figure 2).*

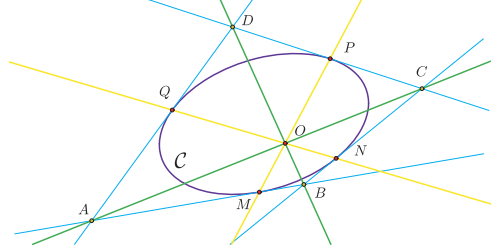


Figure 2: Lemma 2.2

The Lemmas 2.1 and 2.2 will be used to prove the other interesting relations among the lines and points that naturally occur in a quadrilateral inscribed in conics configurations. Many points are going to be introduced so we are going to organize labels of the points.

Let $A_1A_2A_3A_4$ be a quadrilateral inscribed in a conic \mathcal{C} and let M_1 be the intersection point of the lines A_1A_2 and A_3A_4 , M_2 of A_2A_3 and A_4A_1 and M_3 of A_3A_1 and A_2A_4 . Let N_3 be the intersection point of the tangent lines to the conic at A_1 and A_3 , P_3 of the tangents at A_2 and A_4 , N_2 of the tangents at A_1 and A_4 , P_2 of the tangents at A_2 and A_3 , N_1 of the tangents at A_1 and A_2 and P_1 of the tangents at A_3 and A_4 . Let U_1 and U_2 be the points where tangents from M_1 touch \mathcal{C} , and analogously V_1 , V_2 and W_1 , W_2 for the points M_2 and M_3 respectively.

Lemma 2.1 states that the points M_1 , M_2 , N_3 , P_3 are collinear, and also the points M_2 , M_3 , N_1 , P_1 and M_3 , M_1 , N_2 and P_2 . Denote this three lines by m_3 , m_1 and m_2 , respectively. We are going to prove that U_1 and U_2 lie on the line m_3 , V_1 and V_2 on the line m_2 and W_1 and W_2 on m_1 - so that m_1 , m_2 and m_3 are the polar lines of the points M_1 , M_2 and M_3 with respect to \mathcal{C} .

Lemma 2.3. *The points U_1 , M_2 , U_2 and M_3 are collinear.*

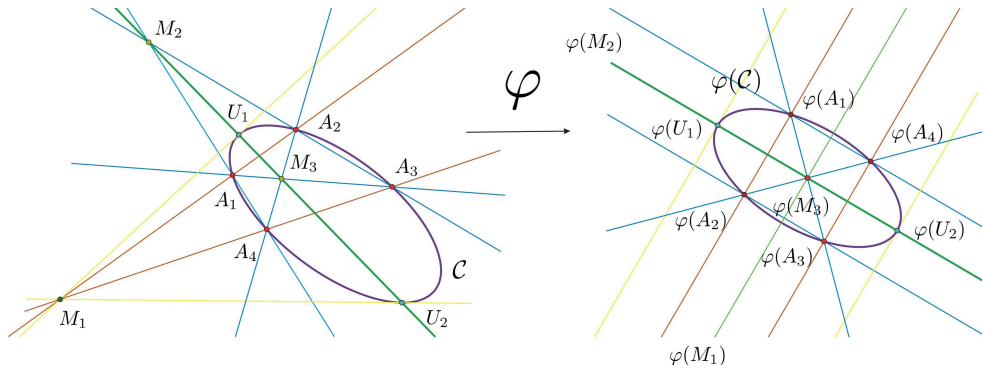


Figure 3: Lemma 2.3

Proof: There is a projective transformation φ that maps the points A_1 , A_2 , A_3 and A_4 onto the vertices of a square. Thus $\varphi(M_3)$ is the center of a square with vertices

$\varphi(A_1)$, $\varphi(A_2)$, $\varphi(A_3)$ and $\varphi(A_4)$. The points $\varphi(M_1)$ and $\varphi(M_2)$ are at infinity. There is a unique way to inscribe the square into the conic, and the lines $\varphi(A_1)\varphi(A_2)$ and $\varphi(A_1)\varphi(A_4)$ are parallel to the axes of the conic $\varphi(\mathcal{C})$. The points $\varphi(U_1)$ and $\varphi(U_2)$ must be mapped onto the axis parallel to the line $\varphi(A_1)\varphi(A_4)$.

Now the points $\varphi(U_1)$, $\varphi(U_2)$, $\varphi(M_2)$ and $\varphi(M_3)$ lie on the axis of conic $\varphi(\mathcal{C})$. Consequently, the points U_1 , M_2 , U_2 and M_3 then lie at the same line. \square

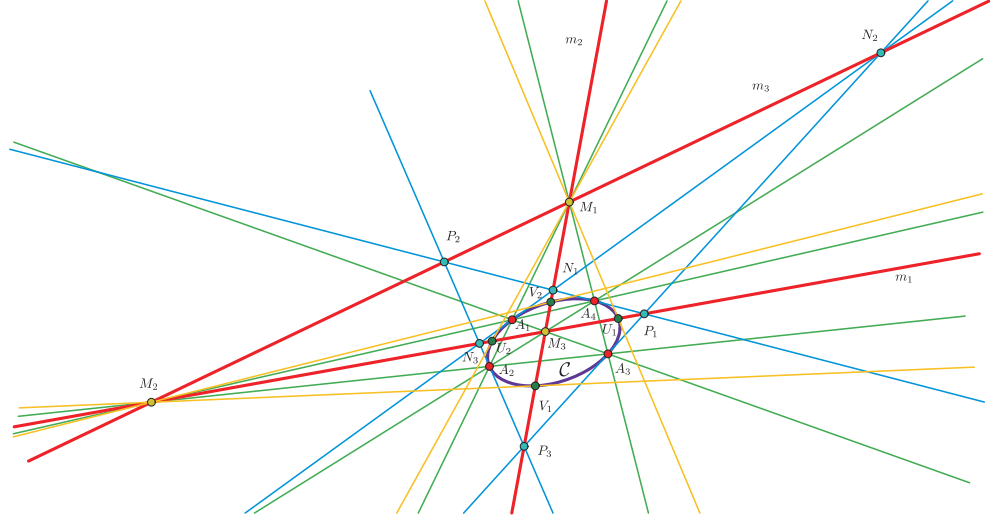


Figure 4: Quadrilateral inscribed in a conic

Lemma 2.3 clearly implies the analogous statement for the lines m_2 and m_3 . This is the classical theorem of the projective geometry and a very useful tool. Some other facts about a quadrilateral inscribed in conics are going to be proved in the next sections.

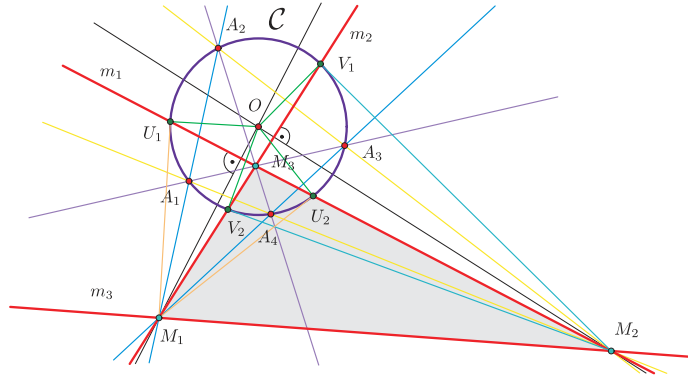


Figure 5: Brocard theorem

In the end, we treat one very special case - when the conic \mathcal{C} is a circle. Projective geometry gives us the plenty of techniques. For example, in the proof of Lemma 2.3 we used the projective transformation. We have already described degeneracy tool when we take some limit cases of polygons inscribed (or circumscribed) in a conic.

It is good to keep in mind that conic could degenerate itself for example to the two lines. This is a way to get interesting configurations of points and lines.

The configuration 4 in the case of a circle has nice a property which is known as the Brocard theorem. Let O be the center of a circle \mathcal{C} . Then the quadrilateral $M_1U_1OU_2$ is deltoid and we get $M_1O \perp m_1$. Similarly, $M_2O \perp m_2$. Thus:

Theorem 2.1 (Brocard theorem). *Let O be the center of circumscribed circle of a cyclic quadrilateral $A_1A_2A_3A_4$. Then O is the orthocenter of triangle $\triangle M_1M_2M_3$.*

3 More lines and pencils of lines

We continue in the same manner. The lines and the pencils of lines we study came from various degenerations of the vertices of hexagon inscribed in a conic. Let us remind that configuration associated with 60 Pascal lines has been described in [13], [23] and [1]. All results from this section could be obtained as the certain degenerate case. But we are going to treat them by elementary means.

Let T_1 be the point of intersection of the line A_3A_4 and tangent at A_1 to \mathcal{C} , T_2 of A_4A_1 and tangent at A_2 , T_3 of A_1A_2 and tangent at A_3 and T_4 of A_2A_3 and tangent at A_4 . Let X_1 be the point of intersection of the line A_2A_3 and tangent at A_1 to \mathcal{C} , X_2 of A_3A_4 and tangent at A_2 , X_3 of A_4A_1 and tangent at A_3 and X_4 of A_1A_2 and tangent at A_4 . Let Y_1 be the point of intersection of the line A_2A_3 and tangent at A_1 , Y_2 of A_1A_4 and tangent at A_2 , Y_3 of A_1A_4 and tangent at A_3 and Y_4 of A_2A_3 and tangent at A_4 .

Proposition 3.1. *The following 16 triples of points are collinear: (M_1, Y_1, Y_2) , (M_1, Y_3, Y_4) , (M_1, X_3, T_4) , (M_1, X_1, T_2) , (M_2, Y_1, Y_4) , (M_2, Y_2, Y_3) , (M_2, X_4, T_1) , (M_2, X_2, T_3) , (M_3, T_1, T_3) , (M_3, X_2, X_4) , (M_3, X_1, X_3) , (M_3, T_2, T_4) , (X_2, Y_3, T_4) , (X_1, Y_2, T_3) , (X_3, Y_4, T_1) , (X_4, Y_1, T_2) .*

Proof: The collinearity of the points M_1 , X_3 and T_4 follows from the Pascal theorem for degenerate hexagon $A_1A_4A_4A_3A_3A_2$, the collinearity of the points M_1 , Y_3 and Y_4 from degenerate hexagon $A_1A_3A_3A_4A_4A_2$ and the collinearity of the points X_2 , Y_3 and T_4 from degenerate hexagon $A_2A_3A_3A_4A_4A_2$. The proof for the rest is analogous. \square

Proposition 3.2. *The following 6 triples of lines are concurrent: (M_2M_3, X_2Y_3, X_3Y_4) , (M_1M_3, X_1Y_2, X_2Y_3) , (M_1M_2, X_1Y_2, X_3Y_4) , (M_2M_3, X_1Y_2, X_4Y_1) , (M_1M_3, X_4Y_1, X_3Y_4) , (M_1M_2, X_4Y_1, X_2Y_3) .*

Proof: By Lemma 4.1 (to be proved in the next section) the points X_1 , X_2 , X_3 , X_4 , T_1 , T_2 , T_3 and T_4 lie on the same conic. From Pascal theorem for the hexagon $T_1X_3X_1T_2X_4X_2$ we get that lines M_1M_3 , X_4Y_1 and X_3Y_4 are concurrent. Analogously for other triples. \square

Define the points as the intersections of the lines: $B_1 = l(A_2V_1) \cap l(A_1V_2)$, $C_1 = l(A_1V_1) \cap l(A_2V_2)$, $D_1 = l(A_3V_1) \cap l(A_4V_2)$, $E_1 = l(A_4V_1) \cap l(A_3V_2)$, $B_3 = l(A_4V_1) \cap l(A_2V_2)$, $C_3 = l(A_4V_2) \cap l(A_2V_1)$, $D_3 = l(A_1V_1) \cap l(A_3V_2)$, $E_3 = l(A_1V_2) \cap l(A_3V_1)$, $D_2 = l(A_4U_1) \cap l(A_1U_2)$, $E_2 = l(A_1U_1) \cap l(A_4U_2)$, $B_2 = l(A_3U_1) \cap l(A_2U_2)$, $C_2 = l(A_2U_1) \cap l(A_3U_2)$, $F_3 = l(A_4U_1) \cap l(A_2U_2)$, $H_3 = l(A_4U_2) \cap l(A_2U_1)$, $G_3 = l(A_1U_1) \cap l(A_3U_2)$, $I_3 = l(A_1U_2) \cap l(A_3U_1)$, $E_1 = l(A_2W_1) \cap l(A_1W_2)$, $F_1 = l(A_1W_1) \cap l(A_2W_2)$, $G_1 = l(A_3W_1) \cap l(A_4W_2)$, $H_1 = l(A_4W_1) \cap l(A_3W_2)$, $H_2 = l(A_4W_1) \cap l(A_1W_2)$, $I_2 = l(A_4W_2) \cap l(A_1W_1)$, $F_2 = l(A_2W_1) \cap l(A_3W_2)$ and $G_2 = l(A_2W_2) \cap l(A_3W_1)$.

4 Surprising conics

In the upper sections many points were introduced. We have showed some of them are collinear while some are the intersections of certain lines. But some of them lie on the curves of degree two!

Lemma 4.1. *The points $X_1, X_2, X_3, X_4, T_1, T_2, T_3$ and T_4 lie on the same conic C_1 ; $Y_1, Y_2, Y_3, Y_4, X_1, X_3, T_2$, and T_4 lie on the same conic C_2 ; $T_1, T_3, X_2, X_4, Y_1, Y_2, Y_3$ and Y_4 lie on the same conic C_3 (see Figure 8).*

Proof: This statement is the special case of the Mystic Octagon theorem, the first time formulated by Wilkinson in [24]. The first conic appears when we consider degenerate octagon $A_1A_2A_2A_3A_3A_4A_4A_1$, the second for $A_1A_3A_3A_2A_2A_4A_4A_1$, and the third for $A_1A_3A_3A_4A_4A_2A_2A_1$. \square

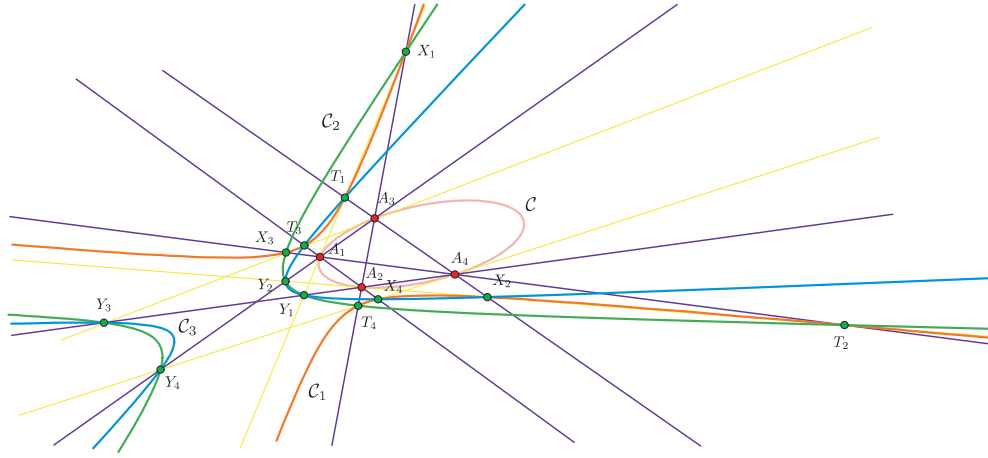


Figure 8: Propositions 4.1

Let J_{2i-1} be the intersection points of the tangents at X_{i-2} and T_i on the conic C_1 , and J_{2i} the intersection points of the tangents at X_{i-1} and T_i (modulo 4), for $i = 1, 2, 3, 4$. Then the following claim is true:

- Theorem 4.1.** • *The lines J_iJ_{i+4} , for $i = 1, 2, 3, 4$ intersect at the point M_3 .*
- *The lines J_1J_7 , J_2J_6 and J_3J_5 intersect at M_1 and the lines J_1J_3 , J_4J_8 and J_5J_7 intersect at M_2 .*
 - *The lines J_1J_4 and J_2J_5 intersect at A_1 , the lines J_4J_7 and J_3J_6 at A_2 , the lines J_6J_1 and J_5J_8 at A_3 and the lines J_3J_8 and J_2J_7 at A_4 .*
 - *The intersection points $l(J_2J_4) \cap l(J_6J_8)$, $l(J_2J_8) \cap l(J_4J_6)$, $l(J_3J_6) \cap l(J_2J_7)$, $l(J_5J_8) \cap l(J_1J_4)$, $l(J_3J_8) \cap l(J_4J_7)$, $l(J_2J_5) \cap l(J_1J_6)$ and $l(J_iJ_{i+1}) \cap l(J_{i+4}J_{i+5})$ for $i = 1, 2, 3, 4$ lie on the same line M_1M_2 .*
 - *The intersection points $l(J_4J_5) \cap l(J_7J_8)$ and $l(J_3J_4) \cap l(J_1J_8)$ lie on the same line M_1M_3 , the intersection points $l(J_2J_3) \cap l(J_5J_6)$ and $l(J_1J_2) \cap l(J_6J_7)$ lie on the same line M_2M_3 .*
 - *The point P_3 lies on the line J_3J_7 and the point N_3 on the line J_1J_5 .*

- Three lines $J_{2i}J_{2i+4}$, $J_{2i+1}J_{2i-2}$ and $J_{2i-1}J_{2i+2}$ (modulo 8) are concurrent for $i = 1, 2, 3, 4$.

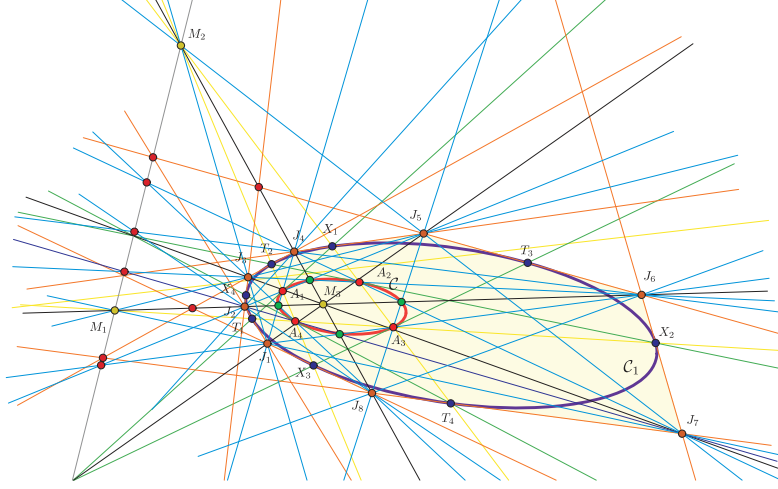


Figure 9: Theorem 4.1

Proof: Consider the quadrilateral formed by tangents to C_1 at J_2 and J_6 . By Lemma 2.2 and Proposition 3.1 the points M_3 and M_2 lie on the line J_2J_6 (we could take the order of points differently). Analogously, the lines J_1J_5 , J_3J_7 and J_4J_8 pass through the point M_3 . In similar fashion we prove other statements for the points M_1 and M_2 , as well as the points N_3 and P_3 .

Lemma 2.2 applied to the quadrilateral formed by tangents to C_1 at J_2 and J_5 proves that line J_2J_5 pass through A_1 . Similarly, A_1 belongs to the line J_1J_4 . Analogously, we prove the corresponding statements for the points A_2 , A_3 and A_4 .

From Lemma 2.1 applied on the quadrilateral $T_2X_1T_4X_3$ and Proposition 3.1, it follows that the intersection point of the lines J_3J_4 and J_7J_8 and the intersection point of the lines J_4J_5 and J_8J_1 lie on the line M_1M_2 . Then by Brianchon Theorem for the hexagon formed by the tangents to C_1 at T_2 , X_1 , T_3 , T_1 , X_3 and T_4 the intersection point of the line J_1J_4 and J_5J_8 lie on the line M_1M_2 . Analogously for the others.

Brianchon Theorem for the hexagon formed by the tangents to C_1 at T_2 , X_1 , X_4 , T_1 , X_3 and T_4 applies the concurrency of the lines J_2J_6 , J_1J_4 and J_5J_8 . We use the similar argument for the rest of the proof. \square

Let K_i be the intersection points of lines J_iJ_{i+1} and $J_{i+2}J_{i+3}$ (modulo 8) for $i = 1, \dots, 8$.

Theorem 4.2. *The points K_i lie on the same conic \mathcal{D}_1 .*

Proof: It is not hard to prove that the lines K_1K_5 , K_2K_6 , K_3K_7 and K_4K_8 pass through the point M_3 , the lines K_2K_3 , K_1K_4 , K_5K_8 and K_6K_7 pass through the point M_1 and the lines K_2K_7 , K_1K_8 , K_3K_6 and K_4K_5 pass through the point M_2 . From the collinearity of the points M_1 , J_2 and $l(J_4J_5) \cap l(J_7J_8)$ the points K_1 , K_2 , K_4 , K_5 , K_7 and K_8 lie on the same conic. Using the similar argument we show that K_2 , K_4 , K_5 , K_6 , K_7 and K_8 lie on the same conic. Because there is a unique conic determined by its 5 points then all the points K_1 , K_2 , K_4 , K_5 , K_6 , K_7 and K_8 are on the same conic. Then it is easy to prove that K_3 also lies on the conic. \square

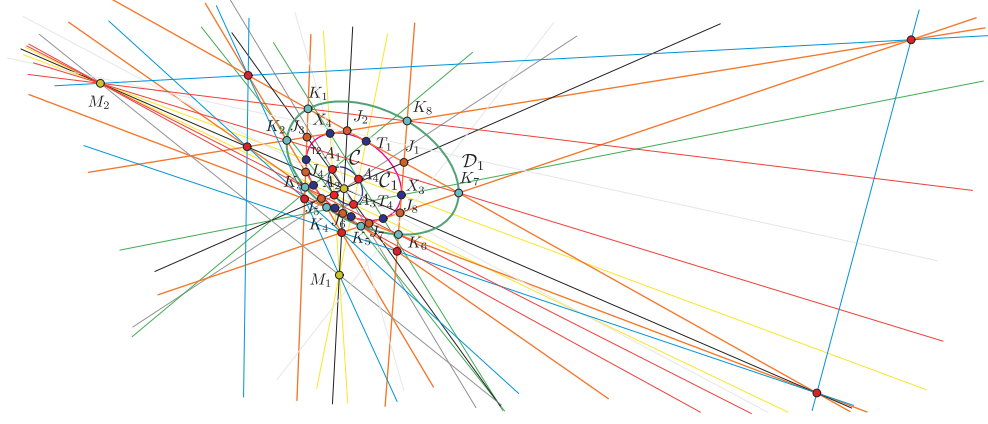


Figure 10: Theorem 4.2

Let $Z_1 = l(M_1U_1) \cap l(M_2V_1)$, $Z_2 = l(M_1U_1) \cap l(M_2V_2)$, $Z_3 = l(M_1U_2) \cap l(M_2V_2)$ and $Z_4 = l(M_1U_2) \cap l(M_2V_1)$.

Theorem 4.3. *The points $N_1, N_2, P_1, P_2, Z_1, Z_2, Z_3$ and Z_4 lie on the same conic.*

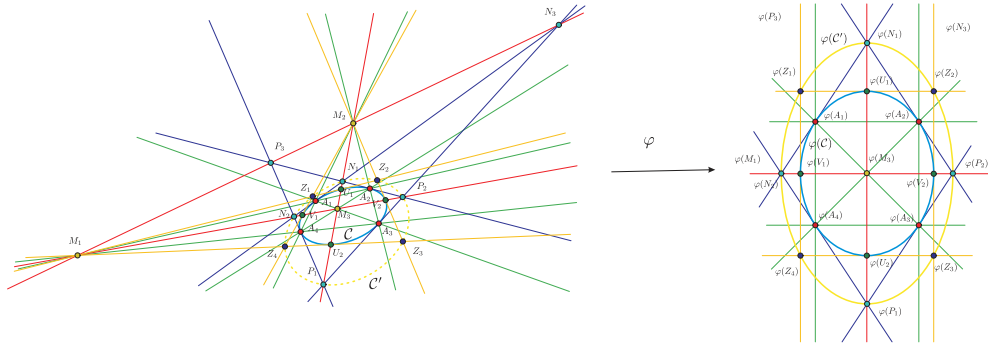


Figure 11: Theorem 4.3

Proof: There exists projective transformation φ that maps vertices A_1, A_2, A_3 and A_4 onto the vertices of a square. Then point $\varphi(M_3)$ is mapped onto the center of conic $\varphi(\mathcal{C})$ and the lines $\varphi(N_1)\varphi(P_1)$ and $\varphi(N_2)\varphi(P_2)$ are the axes. The points $\varphi(U_1)$, $\varphi(U_2)$, $\varphi(V_1)$ and $\varphi(V_2)$ also lie on the axes. As we could see in Figure 11, everything is symmetric and it is easy to conclude that there is a conic through $\varphi(Z_1)$, $\varphi(Z_2)$, $\varphi(Z_3)$, $\varphi(Z_4)$, $\varphi(N_1)$, $\varphi(N_2)$, $\varphi(P_1)$ and $\varphi(P_2)$. \square

Theorems 4.1, 4.2 and 4.3 associate new conics to the quadrilateral inscribed in a conic. They have interesting properties which will be explained in the next section.

5 Poncelet's quadrilateral porism

Jean-Victor Poncelet's famous *Closure theorem* states that if there exists one n -gon inscribed in conic \mathcal{C} and circumscribed about conic \mathcal{D} then any point on \mathcal{C} is the

vertex of some n -gon inscribed in conic \mathcal{C} and circumscribed about conic \mathcal{D} . Poncelet published his theorem in [17]. However, this result influenced mathematics until nowadays. In recent book [7] by Dragovic and Radnovic there are several proofs of Closure theorem, it's generalizations as well as it's relations with elliptic functions theory. The proof is not elementary for general n , although in the case $n = 3$ elegant proof could be found in almost every monograph in projective geometry, see [16].

Theorems 4.2 and 4.3 are the special cases of Poncelet theorem for $n = 4$. Actually, quadrilaterals and conics in them have poristic property. We kept the spirit of elementarity through our paper and our agenda was: Firstlu, we experiment in Cinderella, after that the proof is recovered by elementary tools (again directly guided by Cinderella's tools). In the same style we continue and offer direct analytic proof of Poncelet theorem for quadrilaterals without using differentials and elliptic functions.

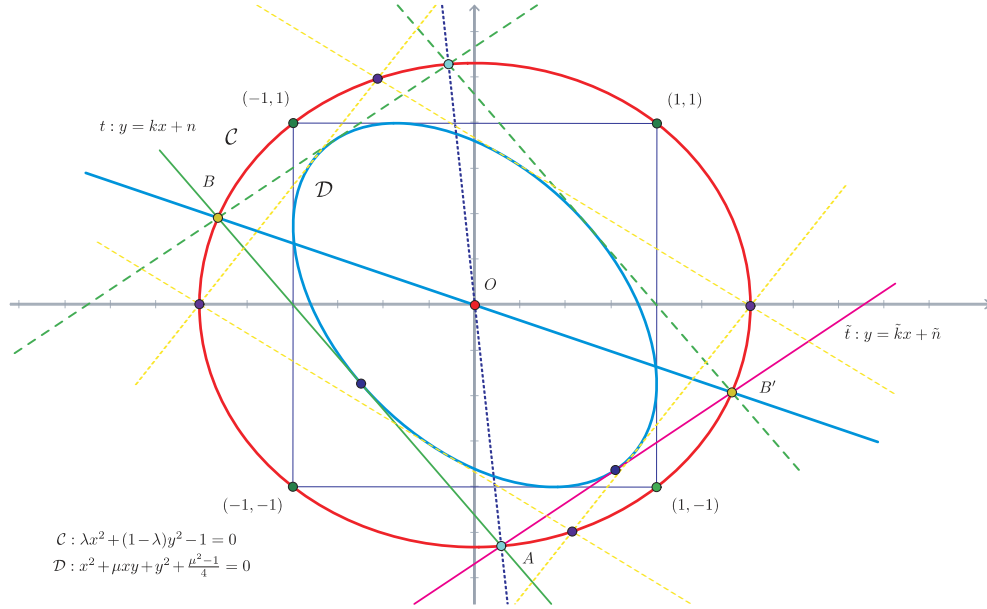


Figure 12: Lemma 5.1 and Theorem 5.1

Lemma 5.1. *Let λ, μ be such that conics $\mathcal{C} : \lambda x^2 + (1 - \lambda)y^2 - 1 = 0$ and $\mathcal{D} : x^2 + \mu xy + y^2 + \frac{\mu^2 - 1}{4} = 0$ are non-degenerate. Let A be a point on \mathcal{C} and B and B' be the intersections of the tangent lines from A to \mathcal{D} with conic \mathcal{C} . Then the points B and B' are symmetric with respect to the origin.*

Proof: Let a line $t : y = kx + n$ be a tangent line to conic \mathcal{D} . The condition of tangency between t and \mathcal{D} is

$$n^2 = k^2 + mk + 1. \quad (1)$$

The coordinates of the intersection points of t and \mathcal{C} are

$$(x_1, y_1) = \left(\frac{-2(1-\lambda)kn - \sqrt{D}}{2(\lambda + (1-\lambda)k^2)}, k \cdot \left(\frac{-2(1-\lambda)kn - \sqrt{D}}{2(\lambda + (1-\lambda)k^2)} \right) + n \right)$$

and

$$(x_2, y_2) = \left(\frac{-2(1-\lambda)kn + \sqrt{D}}{2(\lambda + (1-\lambda)k^2)}, k \cdot \left(\frac{-2(1-\lambda)kn + \sqrt{D}}{2(\lambda + (1-\lambda)k^2)} \right) + n \right),$$

where $D = 4(\lambda - \lambda(1-\lambda)n^2 + (1-\lambda)k^2)$. It is necessary and enough to prove that the line through the points $(-x_1, -y_1)$ and (x_2, y_2) is tangent to \mathcal{D} . This line has the equation $y = \tilde{k}x + \tilde{n}$ where \tilde{k} and \tilde{n} could be calculated as

$$\tilde{k} = \frac{-\lambda}{(1-\lambda)k} \text{ and } \tilde{n} = \frac{\sqrt{D}}{2k(1-\lambda)}. \quad (2)$$

We need to check if

$$\tilde{n}^2 = \tilde{k}^2 + m\tilde{k} + 1.$$

It is directly verified that condition (1) multiplied by $\frac{\lambda(1-\lambda)}{k^2(1-\lambda)^2}$ finishes our proof. \square

Theorem 5.1. *Let \mathcal{C} and \mathcal{D} be conics such that there exists one quadrilateral inscribed in a conic \mathcal{C} and circumscribed about a conic \mathcal{D} . Then any point on \mathcal{C} is the vertex of some quadrilateral inscribed in conic \mathcal{C} and circumscribed about conic \mathcal{D} .*

Proof: There exists a projective transformation that maps the vertices of the quadrilateral inscribed in conic \mathcal{C} and circumscribed about conic \mathcal{D} onto the points $(1, 1)$, $(1, -1)$, $(-1, -1)$ and $(-1, 1)$ (in the standard chart). Thus, conics \mathcal{C} and \mathcal{D} are transformed in those with the equations as in Lemma 5.1. Now the claim follows. \square

In fact, we proved more. All quadrilaterals with poristic property with respect to \mathcal{C} and \mathcal{D} have the common point of the intersection of diagonals (lines joining opposite vertices) and the common line passing through the intersections of opposite side lines. Our work in previous section, now could be reviewed in the new light.

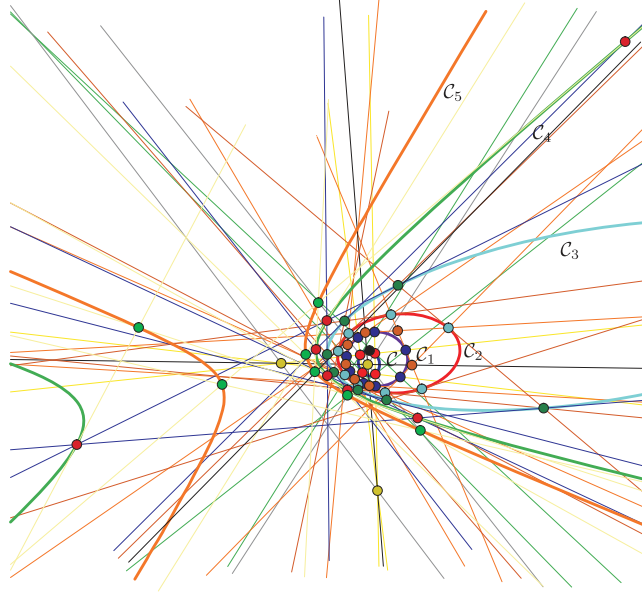


Figure 13: The first five conics in the sequence

Theorems 4.1, 4.2 i 4.3 are obtained after we defined certain points. If we apply the same procedure for defining new points on the points and conics in Theorems,

again we come to similar conclusions. Thus, by repeating this procedure, we obtain the infinite sequence of conics, see Figure 13. Every two consecutive conics in this sequence are Poncelet 4-connected.

Our theorems resemble Darboux's theorem, see [6]. They could be seen as a very special case of Dragović-Radnović theorem 8.38, [7]. Such constructions are also studied in the paper of Schwartz, see [20]. The following result further explains their connection, but first we define 16 points of the intersections $R_1 = l(Z_1Z_2) \cap l(N_1N_2)$, $R_2 = l(Z_1Z_2) \cap l(N_1P_2)$, $R_3 = l(Z_2Z_3) \cap l(N_1P_2)$, $R_4 = l(Z_2Z_3) \cap l(P_1P_2)$, $R_5 = l(Z_3Z_4) \cap l(P_1P_2)$, $R_6 = l(Z_3Z_4) \cap l(P_1N_2)$, $R_7 = l(Z_1Z_4) \cap l(P_1N_2)$, $R_8 = l(Z_1Z_4) \cap l(N_1N_2)$, $R_9 = l(Z_1Z_2) \cap l(P_1P_2)$, $R_{10} = l(Z_3Z_4) \cap l(N_1P_2)$, $R_{11} = l(Z_2Z_3) \cap l(P_1N_2)$, $R_{12} = l(Z_1Z_4) \cap l(P_1P_2)$, $R_{13} = l(Z_3Z_4) \cap l(N_1N_2)$, $R_{14} = l(Z_1Z_2) \cap l(P_1N_2)$, $R_{15} = l(Z_1Z_4) \cap l(N_1P_2)$ and $R_{16} = l(Z_2Z_3) \cap l(N_1N_2)$, see Figure 14.

Theorem 5.2. *The next groups of 8 points lie on the same conic:*

$\{R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8\}$, $\{R_9, R_{10}, R_{11}, R_{12}, R_{13}, R_{14}, R_{15}, R_{16}\}$,
 $\{R_1, R_2, R_5, R_6, R_{11}, R_{12}, R_{15}, R_{16}\}$, $\{R_3, R_4, R_7, R_8, R_9, R_{10}, R_{13}, R_{14}\}$,
 $\{R_1, R_5, R_7, R_9, R_{11}, R_{13}, R_{15}\}$ and $\{R_2, R_3, R_4, R_6, R_8, R_{10}, R_{12}, R_{14}, R_{16}\}$.

The proof of Theorem 5.2 uses the same arguments we used in the previous proofs so we omit it.

If we look at the conic \mathcal{C} and the conic \mathcal{F} through the points $\{R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8\}$ we see they are Poncelet 8-connected and appropriate conics from Theorem 5.2, conic from Theorem 4.3 with the line M_1M_2 form Poncelet-Darboux grid. Two conics $\{R_2, R_3, R_4, R_6, R_8, R_{10}, R_{12}, R_{14}, R_{16}\}$ and $\{R_1, R_5, R_7, R_9, R_{11}, R_{13}, R_{15}\}$ are not coming from Poncelet-Darboux grid, but they could be directly obtained from Dragović-Radnović theorem 8.38, [7]. This result improves the result of Schwartz [20] in a particular case.

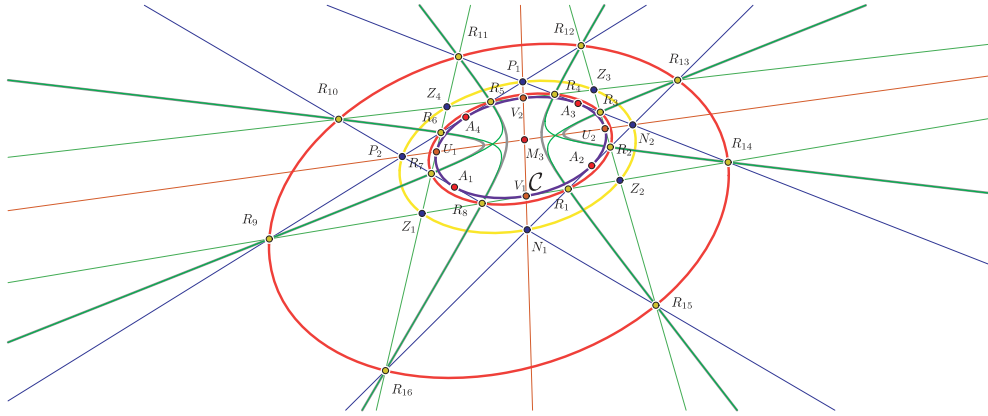


Figure 14: Theorem 5.2

6 Few words about Tabachnikov-Schwartz Theorem 4c

In the end we think it is suitable to say something about already mentioned Theorem 4c stated in [21]. Tabachnikov and Schwartz asked us for the proof. For this occasion

we reformulate it in the following manner:

Theorem 6.1 (Tabachnikov-Schwartz Theorem 4c). *Let $A_1A_2 \dots A_{12}$ be a 12-gon inscribed in a conic \mathcal{C} . Let π maps 12-gon $X_1X_2 \dots X_{12}$ onto a new 12-gon according to the rule $\pi(X_i) = l(X_iX_{i+4}) \cap l(X_{i+1}X_{i+5})$. Then, 12-gon $A_1A_2 \dots A_{12}$ is mapped with $\pi^{(3)} = \pi \circ \pi \circ \pi$ onto a 12-gon inscribed in a conic.*

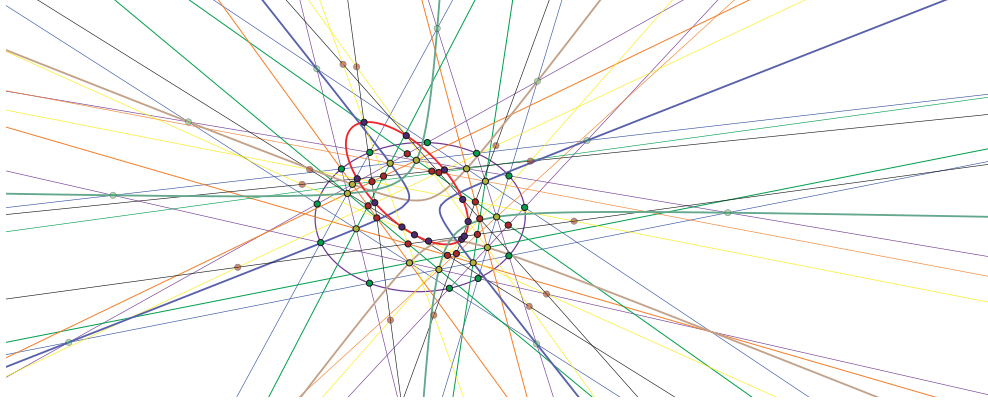


Figure 15: Tabachnikov-Schwartz Theorem 4c

This theorem was the starting point of our research. It seemed that this theorem is a perfect candidate to use the technique illustrated in [1], although in an unpublished paper of Tabachnikov [22] one can find nice proofs for the theorems from [21]. Encouraged by our previous success, we tried to prove Theorem 4c. We used *Cinderella* again to test the result and to obtain a nice picture. But at the beginning we present the problem. We will explain Figure 15 carefully. We start with a 12-gon $A_1A_2 \dots A_{12}$ (the green points lying on the violet conic) inscribed in a conic and define the (yellow) points obtained by π , (blue and violet lines), $\pi^{(2)}$ the red points (green and orange lines) and $\pi^{(3)}$ the violet points (black and yellow lines). It looks like that at the every step we have a 6×6 cage of curves, see [10]. But instead of dealing with 24 points at the second step we take only 12 of them. It is not possible to catch the curves we want in the cage. By Mystic Octagon theorem we could catch three interesting conics and one quartic in the blue-violet cage. What to do with curves at other steps. Definitely we should try to add some new points and then apply Bézout's theorem or a similar statement. But what are that points and how to find them? If we look more carefully, three quadrilaterals can be noticed ($A_1A_4A_7A_{10}$, $A_2A_5A_8A_{11}$ and $A_3A_6A_9A_{12}$) inscribed in a conic and usually the steps are always defined as the certain intersection points of the side lines of quadrilaterals. Thus, we thought if we want to overcome the problems we faced, it is good to understand the quadrilaterals in a conic better.

We have not succeeded in proving the Theorem 4c. But we conducted some experiments in *Cinderella* that we think are important. Firstly, usually algebro-geometric facts give us some freedom (for example, a product of n lines could be often generalized to a curve of degree n , see [1], etc.) but here we have not found any such generalizations. Also, the technique in [1] usually does not differ order of points, that means that certain permutations lead to new objects of the same type (for example Pascal lines). Due to the difference of three quadrilaterals we did not

find new conic at the third step. After all these experiments we believe Tabachnikov and Schwartz Theorem 4c is more surprising and deeper fact then it looks at the first glance!

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